

# Bi-orthogonal Polynomial Sequences and the Asymmetric Simple Exclusion Process

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## Abstract

We reformulate the Corteel-Williams equations for the stationary state of the two parameter Asymmetric Simple Exclusion Process (TASEP) as a linear map  $\mathcal{L}(\cdot)$ , acting on a tensor algebra built from a rank two free module with basis  $\{e_1, e_2\}$ . From this formulation we construct a pair of sequences,  $\{P_n(e_1)\}$  and  $\{Q_m(e_2)\}$ , of bi-orthogonal polynomials (BiOPS), that is, they satisfy  $\mathcal{L}(P_n(e_1) \otimes Q_m(e_2)) = \Lambda_n \delta_{n,m}$ . The existence of the sequences arises from the determinant of a Pascal triangle like matrix of polynomials. The polynomials satisfy first order (uncoupled) recurrence relations. We show that the two first moments  $\mathcal{L}(P_n e_1 Q_m)$  and  $\mathcal{L}(P_n e_2 Q_m)$  give rise to a matrix representation of the ASEP diffusion algebra and hence provide an understanding of the origin of the matrix product Ansatz. The second moment  $\mathcal{L}(P_n e_1 e_2 Q_m)$  defines a tridiagonal matrix which makes the connection with Chebyshev-like orthogonal polynomials.

**Keywords:** bi-orthogonal polynomials, orthogonal polynomials, asymmetric simple exclusion process, LU-decomposition, diffusion algebra

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# 1 Introduction

The two parameter Simple Asymmetric Exclusion Process (also known as the TASEP) is a stochastic process defined by particles hopping along a line of  $L$  sites – see Figure 1. Particles hop on to the line on the left with probability  $\alpha$ , off at the right with probability  $\beta$  and they hop to neighbouring sites to the right with unit probability with the constraint that only one particle can occupy a site. The problem of computing the stationary probability

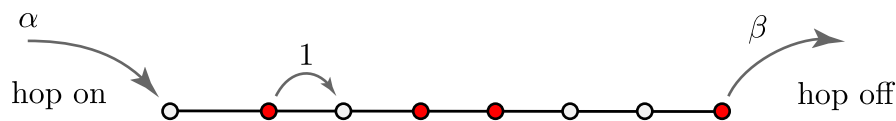


Figure 1: Two parameter ASEP hopping model

distribution was solved by Derrida et. al. [1] with the introduction of the matrix product Ansatz (see below) which provides an algebraic method of computing the stationary distribution. A recent review of the Asymmetric Exclusion Process may be found in Blythe and Evans [2].

As explained in detail below, the matrix product Ansatz expresses the stationary distribution of a given state as a matrix product (the exact form of the product depends on the state). The matrices arise as representations of the ASEP diffusion algebra [1, 3]. The paper by Derrida et al [1] originally found three different representations. As shown by Brak and Essam [4], each matrix representation can be interpreted as a transfer matrix for a different lattice path model. Once the matrix product Ansatz is justified the problem of computing the stationary distribution is thus combinatorially equivalent to finding certain lattice path weight polynomials (and algebraically equivalent to finding a generally expression for certain products of matrices).

The matrix product Ansatz was originally proposed and justified in [1] by showing it satisfied the stationary master equation of the ASEP. A more flexible matrix Ansatz was constructed by Corteel and Williams [5] who produced a set of more general equations the matrices are required to satisfy (Theorem 5.2 in [5]).

The primary objective of this paper is to find a systematic way to produce explicit matrix representations of the diffusion algebra (rather than using some ad hoc method). How this arises is stated in the matrix repre-

sensation Theorem 12. This enables us to interpret the matrix elements as bi-orthogonal moments.

In order to get to the matrix representation theorem we begin by rewriting the Corteel-Williams equations for the two parameter ASEP as a linear map  $\mathcal{L}(\cdot)$  acting on a tensor algebra. This algebra is built from a rank two free module with generators  $\{e_1, e_2\}$ . The linear map  $\mathcal{L}(\cdot)$  acting on the tensor algebra makes clear the intimate connection between linearity (or multi-linearity) and stationary.

From this formulation we construct a pair of sequences,  $\{P_n(e_1)\}_{n \geq 0}$  and  $\{Q_n(e_2)\}_{n \geq 0}$ , of monic polynomials in the generators  $e_1$  and  $e_2$  respectively. These sequences are shown to be “bi-orthogonal” with respect to the linear map  $\mathcal{L}(\cdot)$  in that they satisfy

$$\mathcal{L}(P_n(e_1) \otimes Q_m(e_2)) = \Lambda_n \delta_{n,m}.$$

A pair of sequences of such polynomials orthogonal with respect to a linear form will be called a “bi-orthogonal pair of polynomial sequences” (BiOPS).

The existence (and uniqueness) of the BiOPS arises from the determinant of a matrix of Pascal triangle generated polynomials. The polynomials satisfy first order (uncoupled) recurrence relations and hence have simple product forms. We show that the two first moments  $\mathcal{L}(\hat{P}_n \otimes e_1 \otimes \hat{Q}_m)$  and  $\mathcal{L}(\hat{P}_n \otimes e_2 \otimes \hat{Q}_m)$ , where  $\hat{P}_n$  and  $\hat{Q}_n$  are the normalised versions of  $P_n$  and  $Q_n$ , afford a matrix representation of the ASEP diffusion algebra and hence provide an understanding of the origin of the third representation of Derrida et. al.

The second moment  $\mathcal{L}(P_n e_1 e_2 Q_m)$  defines a tridiagonal matrix which makes the connection between the BiOPS and classical (ie. three term recurrence) Chebyshev-like orthogonal polynomials.

## 2 The Modules

In this section we define the various modules and rings we will use throughout this paper. Let  $\mathcal{R}$  be the ring of integer coefficient commutative polynomials,  $\mathbb{Z}[\alpha, \beta]$  and  $\mathcal{M}$  the  $\mathcal{R}$ -module

$$\mathcal{M} = \bigoplus_{n \geq 0} \mathcal{V}_2^{\otimes n} \tag{2.1}$$

where  $\mathcal{V}_2$  is a free rank two  $\mathcal{R}$ -module with generators  $e_1, e_2$ . Here  $\mathcal{V}^0$  denotes the ring  $\mathcal{R}$  of the module and  $\mathcal{V}_2^{\otimes n} = \mathcal{V}_2 \otimes \mathcal{V}_2 \otimes \cdots \otimes \mathcal{V}_2$  ( $n$  factors).

The submodule  $\mathcal{V}^{\otimes n}$ , of degree  $n$ , is generated by the ‘standard’ basis elements  $e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_n}$  where  $e_i \in \{e_1, e_2\}$ . The module  $\mathcal{M}$  is then given the usual graded tensor algebra structure via the basis product rule  $(e_{i_1} \otimes \cdots \otimes e_{i_n}) \times (e_{j_1} \otimes \cdots \otimes e_{j_m}) = e_{i_1} \cdots \otimes e_{i_n} \otimes e_{j_1} \cdots \otimes e_{j_m}$  and extended linearly to other elements of  $\mathcal{M}$  (see [6] section XVI.5). For brevity we will frequently omit the tensor product symbol, thus  $e_1^m e_2^n$  denotes  $e_1^{\otimes m} \otimes e_2^{\otimes n}$  etc.

The two parameter version of the Corteel-Williams equations given in [5] was stated in terms of matrices and vectors. Here we give a slightly more abstract version by using a linear form (ie. a module homomorphism to  $\mathcal{R}$ ) formulation.

**Definition 1** (Corteel-Williams Equations). *Let  $a, b$  be any standard basis elements of  $\mathcal{M}$ . The  $\mathcal{R}$ -module homomorphism  $\mathcal{L} : \mathcal{M} \rightarrow \mathcal{R}$  is defined by the following equations:*

$$\mathcal{L}(a \otimes e_1 \otimes e_2 \otimes b) = \alpha\beta \mathcal{L}(a \otimes (e_1 + e_2) \otimes b) \quad (2.2a)$$

$$\mathcal{L}(a \otimes e_1) = \alpha \mathcal{L}(a) \quad (2.2b)$$

$$\mathcal{L}(e_2 \otimes b) = \beta \mathcal{L}(b) \quad (2.2c)$$

with  $\mathcal{L}(1) = 1$  and extended linearly to other elements of  $\mathcal{M}$ .

Note, in [1] the equation corresponding to (2.2a) does not contain the  $\alpha\beta$  factor. Thus in [1] the diffusion algebra appears as “ $DE = D + E$ ”, whilst here it is “ $DE = \alpha\beta(D + E)$ ”. This is a small but important difference between the Corteel-Williams equations compared with those given in [1]. The effect of this difference for the two parameter case is that the polynomials in the numerator and denominator of probabilities in [1] are naturally computed in the *reciprocal* variables  $1/\alpha$  and  $1/\beta$ . This does not change the probability as the numerator and denominator can be rewritten in terms of  $\alpha$  and  $\beta$  by removing a common factor (which cancels).

The matrix product Ansatz of [1] can now be restated using the map  $\mathcal{L}$ .

**Theorem 1** (Derrida, Evans, Hakim and Pasquier [1]). *The stationary state probability distribution,  $f(\tau)$ , of the two parameter ASEP for the system in state  $\tau = (\tau_1, \dots, \tau_L)$ , is given by*

$$f(\tau) = \frac{1}{Z_L} \mathcal{L} \left( \prod_{i=1}^L (\tau_i e_1 + (1 - \tau_i) e_2) \right) \quad (2.3)$$

where

$$Z_L = \mathcal{L}\left((e_1 + e_2)^L\right) \quad (2.4)$$

and  $\tau_i = 1$  if site  $i$  is occupied and zero otherwise.

### 3 Bi-Orthogonal Pair of Polynomial Sequences

Consider the pair of sequences,

$$\{P_n(e_1)\}_{n \geq 0}, \quad \{Q_n(e_2)\}_{n \geq 0} \quad (3.1)$$

of monic polynomials (where  $P_n$  and  $Q_n$  are degree  $n$ ). We wish to determine if it is possible to find such a pair which are also orthogonal with respect to  $\mathcal{L}$ . In particular, do there exist such sequences for which  $\mathcal{L}(P_n \otimes Q_m) = \Lambda_m \delta_{n,m}$  (and  $\Lambda_n$  non-zero)?

In order to show they do indeed exist we consider the infinite dimensional ‘bi-moment matrix’,  $\mathbf{B}$ , defined by

$$\mathbf{B}_{n,m} = \mathcal{L}(e_1^n e_2^m), \quad n, m \in \mathbb{N}_0. \quad (3.2)$$

where  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$  and  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Using the Corteel-Williams equations, (2.2), it is then simple to prove that the bi-moment matrix elements satisfy a Pascal triangle like partial difference equation as given in the following theorem.

**Theorem 2.** *Let  $\mathbf{B}$  be the bi-moment matrix defined by  $\mathbf{B}_{n,m} = \mathcal{L}(e_1^n e_2^m)$  with  $n, m \in \mathbb{N}_0$ . Then*

$$\mathbf{B}_{n,m} = \alpha\beta (\mathbf{B}_{n,m-1} + \mathbf{B}_{n-1,m}), \quad n, m \in \mathbb{N}, \quad (3.3)$$

with boundary values  $\mathbf{B}_{n,0} = \alpha^n$  and  $\mathbf{B}_{0,m} = \beta^m$ .

Thus the matrix looks like

$$\mathbf{B} = \begin{pmatrix} 1 & \beta & \beta^2 & \dots \\ \alpha & \alpha\beta(\alpha + \beta) & \alpha\beta^2(\alpha^2 + \alpha\beta + \beta) & \dots \\ \alpha^2 & \alpha^2\beta(\alpha + \alpha\beta + \beta^2) & \alpha^2\beta^2(\alpha^2 + 2\alpha^2\beta + 2\alpha\beta^2 + \beta^2) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The existence of the BiOPS requires that the determinant of the  $(n+1) \times (n+1)$  sub-matrix

$$\mathbf{B}^{(n)} = (\mathbf{B}_{i,j})_{0 \leq i,j \leq n}$$

be non-zero for all  $n \geq 0$ .

This determinant evaluates to a simple form – see Theorem 4 – however it is non-trivial to prove. The determinant of a very similar matrix was stated by Bacher [7] and subsequently proved by Krattenthaler [8] (using LU decomposition). Krattenthaler’s theorem is as follows.

**Theorem 3** (Krattenthaler, [8]). *Let  $(A_{i,j})_{i,j \geq 0}$  be a matrix whose elements are given by the recurrence*

$$A_{i,j} = A_{i-1,j} + A_{i,j-1} + xA_{i-1,j-1} \quad i, j \geq 1 \quad (3.4)$$

*with initial conditions  $A_{i,0} = \rho^i$  and  $A_{0,j} = \sigma^j$ . Then*

$$\det_{0 \leq i,j \leq n-1} (A_{i,j}) = (1+x)^{\binom{n-1}{2}} (x + \rho + \sigma + \rho\sigma)^{n-1}. \quad (3.5)$$

In order to use Theorem 3 to evaluate  $\det \mathbf{B}^{(n)}$ , it first has to be reduced to one whose elements satisfy the recurrence relation (3.4). This can be done using row operations (divide row  $i$  by  $(\alpha\beta)^i$  and column  $j$  by  $(\alpha\beta)^j$ ). The resulting matrix is then  $A_{i,j}$  with  $x = 0$ ,  $\rho = 1/\alpha$  and  $\sigma = 1/\beta$ . Compensating for the row and column operations introduces a factor of  $(\alpha\beta)^{n^2}$ . Thus from Theorem 3 we get the following.

**Theorem 4.** *Let  $\mathbf{B}^{(n)} = (\mathbf{B}_{i,j})_{0 \leq i,j \leq n}$  be the truncated  $(n+1) \times (n+1)$  bi-moment matrix whose elements are defined by Theorem 2. Then*

$$\det \mathbf{B}^{(n)} = (\alpha\beta)^{n^2} (\alpha + \beta - 1)^n. \quad (3.6)$$

For  $n, m \geq 0$  require the bi-orthogonality condition

$$\mathcal{L}(P_n \otimes Q_m) = \Lambda_n \delta_{n,m} \quad (3.7)$$

where  $\Lambda_n$  is a sequence of non-zero normalisation factors (determined by  $\mathcal{L}$ ). We then have the following theorem that shows a unique pair of sequences of such polynomials exists.

**Theorem 5.** *Let  $\{P_n\}_{n \geq 0}$  and  $\{Q_n\}_{n \geq 0}$  be a pair of sequences of monic polynomials satisfying*

$$\mathcal{L}(P_n \otimes Q_m) = \Lambda_n \delta_{n,m} \quad (3.8)$$

*where the map  $\mathcal{L}$  is defined by the Corteel-Williams equations (2.2). Then  $\{P_n\}_{n \geq 0}$  and  $\{Q_n\}_{n \geq 0}$  are unique and are given by*

$$P_n(e_1) = (e_1 - \alpha)(e_1 - \alpha\beta)^{n-1}, \quad n > 0 \quad (3.9a)$$

$$Q_n(e_2) = (e_2 - \beta)(e_2 - \alpha\beta)^{n-1}, \quad n > 0 \quad (3.9b)$$

with  $P_0 = Q_0 = 1$ ,

$$\Lambda_n = (\alpha\beta)^{2n-1}(\alpha + \beta - 1), \quad n > 0 \quad (3.10)$$

and  $\Lambda_0 = 1$ .

*Proof.* First consider  $\{P_n\}$ . The existence of  $\{P_n\}$  follows by applying Cramer's rule to the system of linear equations obtained by writing

$$P_n(e_1) = \sum_{k=0}^n a_k^{(n)} e_1^k \quad (3.11)$$

with  $a_n^{(n)} = 1$  and

$$\mathcal{L}(P_n e_2^k) = \Lambda_n \delta_{n,k}, \quad k \leq n. \quad (3.12)$$

Since  $e_2^k = Q_k(e_2) + \sum_{\ell=1}^k c_\ell^{(k)} Q_\ell(e_2)$  (ie.  $\{Q_n\}$  is a basis set for any monic polynomial in  $e_2$ ) and using equations (3.2) and (3.7) we get the system

$$(a_0^{(n)}, a_1^{(n)}, \dots, a_n^{(n)}) \begin{pmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1} & \dots & \mathbf{B}_{0,n} \\ \mathbf{B}_{1,0} & \mathbf{B}_{1,1} & \dots & \mathbf{B}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n,0} & \mathbf{B}_{n,1} & \dots & \mathbf{B}_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Lambda_n \end{pmatrix}. \quad (3.13)$$

Since for all  $n \geq 0$  we have from Theorem 4 that  $\det \mathbf{B}^{(n)} \neq 0$  and thus the system has a unique solution given by Cramer's rule

$$P_n(e_1) = \frac{1}{\det \mathbf{B}^{(n-1)}} \det \begin{pmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1} & \dots & \mathbf{B}_{0,n-1} & 1 \\ \mathbf{B}_{1,0} & \mathbf{B}_{1,1} & \dots & \mathbf{B}_{1,n-1} & e_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{B}_{n,0} & \mathbf{B}_{n,1} & \dots & \mathbf{B}_{n,n-1} & e_1^n \end{pmatrix}. \quad (3.14)$$

Similarly for  $\{Q_n\}$ ,

$$Q_n(e_2) = \frac{1}{\det \mathbf{B}^{(n-1)}} \det \begin{pmatrix} \mathbf{B}_{0,0} & \mathbf{B}_{0,1} & \dots & \mathbf{B}_{0,n} \\ \mathbf{B}_{1,0} & \mathbf{B}_{1,1} & \dots & \mathbf{B}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{n-1,0} & \mathbf{B}_{n-1,1} & \dots & \mathbf{B}_{n-1,n} \\ 1 & e_2 & \ddots & e_2^n \end{pmatrix}. \quad (3.15)$$

To evaluate the determinants (3.14) and (3.15) the same LU decomposition used by Krattenthaler is applicable. The only relevant change introduced by replacing the final row (or column) by  $1, e_2, \dots, e_2^n$  is to change the  $U_{n,n}$  matrix element of the  $U$  matrix from  $(\alpha + \beta - 1)$  to  $(e_1 - \alpha)(e_1 - \alpha\beta)^{n-1}$  from which (3.9a) trivially follows. A similar argument gives (3.9b). The requirement that the polynomials  $P_n$  and  $Q_n$  are monic fixes the value of  $\Lambda_n$  to

$$\Lambda_n = \frac{\det \mathbf{B}^{(n)}}{\det \mathbf{B}^{(n-1)}}$$

which evaluates to (3.10). □

Note (3.14) and (3.15) are very similar to those obtained when computing classical orthogonal polynomials but have the important difference that the moment matrix (replaced here by the bi-moment matrix) is no longer a Hankel matrix.

On occasion it is more convenient to work with normalised sequences  $\{\hat{P}_n\}$ ,  $\{\hat{Q}_n\}$  of the two polynomials sequences. If (3.7) is replaced by

$$\mathcal{L}(\hat{P}_n \otimes \hat{Q}_m) = \delta_{n,m}, \quad (3.16)$$

then clearly

$$\hat{P}_n = P_n / \sqrt{\Lambda_n}, \quad (3.17a)$$

$$\hat{Q}_n = Q_n / \sqrt{\Lambda_n}. \quad (3.17b)$$

gives a bi-orthonormal pair of polynomial sequences.

Note, for the normalised polynomials the ring  $\mathcal{R}$  is no longer  $\mathbb{Z}[\alpha, \beta]$  but has to be extended to an appropriate algebraic ring of functions of  $\alpha$  and  $\beta$ . All the theorems are readily extended to accommodate this generalisation.

Having obtained explicit expressions for the polynomials it is clear that they satisfy *first order* recurrence relations.

**Theorem 6.** *Let  $P_n$  and  $Q_n$  be the polynomials of Theorem 5. Then the polynomials satisfy the first order recurrences*

$$P_{n+1} = (e_1 - \alpha\beta) \otimes P_n, \quad n \in \mathbb{N} \quad (3.18)$$

$$Q_{n+1} = (e_2 - \alpha\beta) \otimes Q_n, \quad n \in \mathbb{N}. \quad (3.19)$$

with initial values  $P_1 = e_1 - \alpha$  and  $Q_1 = e_2 - \beta$ .



The recurrence relations are equivalent to a certain choice of matrices  $\bar{X}$  and  $\bar{Y}$  in the following product result.

**Theorem 7.** *Let  $P_n$  and  $Q_n$  be the polynomials of Theorem 5. Then*

$$P_n(e_1) \otimes e_1 = \sum_{k=0}^{n+1} \bar{X}_{n,k} P_k(e_1), \quad n > 0 \quad (3.20)$$

and

$$e_2 \otimes Q_n(e_2) = \sum_{k=0}^{n+1} Q_k(e_2) \bar{Y}_{k,n}, \quad n > 0. \quad (3.21)$$

where

$$\bar{X}_{n,m} = \mathcal{L}(P_n e_1 Q_m) / \Lambda_m \quad (3.22)$$

and

$$\bar{Y}_{n,m} = \mathcal{L}(P_n e_2 Q_m) / \Lambda_n. \quad (3.23)$$

*Proof.* Since  $P_n(e_1) e_1$  is a monic polynomial of degree  $n+1$  and  $\{P_n\}_{n \geq 0}$  are a basis set for all monic polynomials (in  $e_1$ ) we have that  $P_n e_1 = \sum_{k=0}^{n+1} d_k^{(n)} P_k$  with  $d_k^{(n)} \in \mathcal{R}$ . Thus

$$\mathcal{L}(P_n e_1 Q_m) = \sum_{k=0}^{n+1} d_k^{(n)} \mathcal{L}(P_k Q_m) = d_m^{(n)} \Lambda_m, \quad m \leq n+1$$

and hence  $d_m^{(n)} = \bar{X}_{n,m}$  which gives (3.22). Similarly for (3.23).  $\square$

Thus, if  $\mathcal{L}(P_n e_1 Q_m)$  and  $\mathcal{L}(P_n e_2 Q_m)$  can be computed (or are given) then the recurrences relations for  $P_n$  and  $Q_n$  are determined via (3.20) and (3.21). Since we already have the recurrence relation from the explicit form of  $P_n$  and  $Q_n$  we can use it to compute the moments. Thus, combining (3.22) and (3.18) (and (3.19) with (3.23)) gives the following result.

**Theorem 8.** *Let  $P_n$  and  $Q_n$  be the polynomials of Theorem 5. The two first moments*

$$X_{n,m} = \mathcal{L}(P_n e_1 Q_m), \quad n, m \in \mathbb{N}_0 \quad (3.24)$$

$$Y_{n,m} = \mathcal{L}(P_n e_2 Q_m), \quad n, m \in \mathbb{N}_0 \quad (3.25)$$

are given by

$$X_{n,m} = \begin{cases} \alpha\Lambda_m & n = m = 0 \\ \alpha\beta\Lambda_m & 0 < m = n \\ \Lambda_m & m = n + 1 \end{cases} \quad (3.26)$$

$$Y_{n,m} = \begin{cases} \beta\Lambda_n & n = m = 0 \\ \alpha\beta\Lambda_n & 0 < m = n \\ \Lambda_n & m + 1 = n \end{cases} \quad (3.27)$$

Thus the column (respectively, row) scaled matrices

$$\bar{X}_{n,m} = X_{n,m} \frac{1}{\Lambda_m} \quad (3.28)$$

$$(3.29)$$

$$\bar{Y}_{n,m} = \frac{1}{\Lambda_n} Y_{n,m} \quad (3.30)$$

have the bi-diagonal forms

$$\bar{X} = \begin{pmatrix} \alpha & 1 & & & \\ & \alpha\beta & 1 & & \\ & & \alpha\beta & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \quad (3.31)$$

and

$$\bar{Y} = \begin{pmatrix} \beta & & & & \\ 1 & \alpha\beta & & & \\ & 1 & \alpha\beta & & \\ & & 1 & \alpha\beta & \\ & & & \ddots & \ddots \end{pmatrix} \quad (3.32)$$

where the zero elements are omitted for clarity.

If the bi-orthonormal polynomials, (3.17), are used, then with

$$\hat{X}_{n,m} = \mathcal{L}(\hat{P}_n e_1 \hat{Q}_m), \quad n, m \in \mathbb{N}_0 \quad (3.33)$$

$$\hat{Y}_{n,m} = \mathcal{L}(\hat{P}_n e_2 \hat{Q}_m), \quad n, m \in \mathbb{N}_0 \quad (3.34)$$

we get the bi-diagonal matrices

$$\hat{X} = \begin{pmatrix} \alpha & \sqrt{\alpha\beta(\alpha+\beta-1)} & & & \\ & \alpha\beta & \alpha\beta & & \\ & & \alpha\beta & \alpha\beta & \\ & & & \ddots & \ddots \end{pmatrix} \quad (3.35)$$

and

$$\hat{Y} = \begin{pmatrix} \beta & & & & \\ \sqrt{\alpha\beta(\alpha+\beta-1)} & \alpha\beta & & & \\ & \alpha\beta & \alpha\beta & & \\ & & \alpha\beta & \alpha\beta & \\ & & & \ddots & \ddots \end{pmatrix}. \quad (3.36)$$

## 4 Shock Ring and Boundary Modules

In this section we briefly address the question of the algebraic status of the bi-orthogonal polynomials. As will be seen below, it turns out they form factors for the left and right boundary quotient modules.

The three Corteel-Williams equations motivate the study of three cyclic  $\mathcal{R}$ -submodules of  $\mathcal{M}$  each generated by one of the relations

$$e_1 \otimes e_2 - \alpha\beta(e_1 + e_2), \quad (4.1a)$$

$$e_1 - \alpha, \quad (4.1b)$$

$$e_2 - \beta. \quad (4.1c)$$

Thus there is the two sided principal ideal

$$\mathcal{I} = \langle e_1 \otimes e_2 - \alpha\beta(e_1 + e_2) \rangle, \quad (4.2)$$

a cyclic left submodule

$$J_L = \langle e_1 - \alpha \rangle_{\text{left}}, \quad (4.3)$$

and a cyclic right submodule

$$J_R = \langle e_2 - \beta \rangle_{\text{right}}. \quad (4.4)$$

The ideal  $\mathcal{I}$  will give us a quotient ring which will lead to a matrix representation. The two cyclic submodules are one-sided ideals however since they

are not two-sided they do not lead to quotient rings but will give quotient modules associated with the bi-orthogonal polynomials.

In order to characterise the various quotient structures obtained from  $\mathcal{I}$ ,  $J_L$  and  $J_R$  we define three  $\mathcal{R}$ -modules. First, let

$$J_L = \left\{ \sum_i^* p_i (e_1 - \alpha) : p \in \mathcal{M} \right\} \quad (4.5)$$

and

$$J_R = \left\{ \sum_i^* (e_2 - \beta) p_i : p \in \mathcal{M} \right\} \quad (4.6)$$

be  $\mathcal{R}$ -submodules of  $\mathcal{M}$  where  $\sum_i^*$  denotes a finite sum. Then the three  $\mathcal{R}$ -modules are

$$\mathcal{S} = \bigoplus_{n,m \in \mathbb{N}_0} \mathcal{R} e_2^n \otimes e_1^m, \quad (4.7a)$$

$$\mathcal{B}_r = \mathcal{R} + \bigoplus_{n \in \mathbb{N}} \mathcal{M}_1 e_2^n, \quad (4.7b)$$

$$\mathcal{B}_l = \mathcal{R} + \bigoplus_{n \in \mathbb{N}} e_1^n \mathcal{M}_2. \quad (4.7c)$$

Thus,  $\mathcal{S}$  is all finite  $\mathcal{R}$ -linear combinations of the “normal ordered” elements  $e_2^n \otimes e_1^m$ ,  $\mathcal{B}_r$  is all finite  $\mathcal{R}$ -linear combinations of elements of  $\mathcal{M}$  whose rightmost factor is  $e_2^n$  (for some  $n > 0$ ) and  $\mathcal{B}_l$  is all finite  $\mathcal{R}$ -linear combinations of elements of  $\mathcal{M}$  whose leftmost factor is  $e_1^n$  (for some  $n > 0$ ). We will refer to  $\mathcal{B}_r$  and  $\mathcal{B}_l$  as the “boundary” modules and  $\mathcal{S}$  as the “shock” module (since the particle state corresponding to  $e_2^n \otimes e_1^m$  corresponds to  $n$  empty sites followed by  $m$  occupied sites and thus resembles a shock state).

The shock module  $\mathcal{S}$  is given a ring structure with a product,  $\bullet : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$  defined recursively using (4.1a),

$$e_2^n e_1^m \bullet e_2^k e_1^\ell = \alpha \beta e_2^n e_1^m \bullet e_2^{k-1} e_1^\ell + \alpha \beta e_2^n e_1^{m-1} \bullet e_2^k e_1^\ell. \quad (4.8)$$

and extended linearly to other elements of  $\mathcal{S}$ . Note, this is the same recurrence relation that occurred in defining the bi-moment matrix.

Since  $\mathcal{I}$  is a two-sided ideal of  $\mathcal{M}$  the quotient ring  $\mathcal{M}/\mathcal{I}$  exists. The following theorem characterises the quotient.

**Theorem 9.** *Let  $\mathcal{M}$  be the ring (2.1) and  $\mathcal{I}$  be the ideal given by (4.2). Then we have the following ring isomorphism*

$$\mathcal{M}/\mathcal{I} \cong \mathcal{S}, \quad (4.9)$$

where the shock ring  $\mathcal{S}$  is given by (4.7a) with product rule (4.8).

The proof is a standard proof based on the commuting diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\pi} & \mathcal{M}/\mathcal{I} \\ & \searrow \phi_1 & \downarrow \theta_1 \\ & & \mathcal{S} \end{array}$$

where  $\pi$  is the standard projection map

$$\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{I}. \quad (4.10)$$

The map  $\phi_1$  is that defined by repeated substitution of  $e_1 e_2 = \alpha \beta (e_1 + e_2)$  and  $\theta_1$  is the isomorphism (4.9) (since  $\ker(\phi_1) = \mathcal{I}$ ). We can now see where the bi-orthogonal polynomials enter. Since

$$\mathcal{B}_r = \mathcal{R} + \bigoplus_{n \in \mathbb{N}} \mathcal{M}_1 \otimes e_2^n, \quad (4.11)$$

$$\mathcal{B}_l = \mathcal{R} + \bigoplus_{n \in \mathbb{N}} e_1^n \otimes \mathcal{M}_2. \quad (4.12)$$

the right factors (respec. left) define a sequence of monomials  $\{e_2^n\}_{n \geq 0}$  (respec.  $\{e_1^n\}_{n \geq 0}$ ). However, they can be replaced by *any* sequences of degree  $n$  monic polynomials. Thus, using  $\{P_n(e_1)\}$  and  $\{Q_n(e_2)\}$  gives

$$\mathcal{B}_r = \mathcal{R} + \bigoplus_{n \in \mathbb{N}} \mathcal{M}_1 \otimes Q_n(e_2), \quad (4.13)$$

$$\mathcal{B}_l = \mathcal{R} + \bigoplus_{n \in \mathbb{N}} P_n(e_1) \otimes \mathcal{M}_2. \quad (4.14)$$

We thus see that the bi-orthogonal pair of polynomial sequences form left and right factors for the boundary modules. If we only consider the submodule

$$\mathcal{W} = \bigoplus_{n, m \in \mathbb{N}_0} \mathcal{R} e_1^n \otimes e_2^m$$

then  $\{P_n(e_1) \otimes Q_m(e_2)\}$  form a basis set for  $\mathcal{W}$ . Extending this to other basis elements of  $\mathcal{M}$  (grouped into powers and  $e_1$  and  $e_2$ ) shows that the standard basis elements of  $\mathcal{M}$  can be written as alternating products of  $P_n$ 's and  $Q_m$ 's, for example

$$e_1^{\otimes 5} \otimes e_2^{\otimes 3} \otimes e_1 \otimes e_2^{\otimes 2} \otimes e_1 \mapsto P_5 \otimes Q_3 \otimes P_1 \otimes Q_2 \otimes P_1.$$

giving a new basis set for  $\mathcal{M}$ . However, these basis elements are not orthogonal with respect to  $\mathcal{L}$ . This can be seen by using the matrix representations of  $P_n$  and  $Q_m$  – see next section.

## 5 A Matrix Representation of the Shock Ring

Let  $\text{Mat}_d$  be the ring of  $d$  dimensional matrices (where  $d$  may be infinite) with entries from  $\mathcal{R}$ . We seek representations, that is, ring homomorphisms, of the shock ring to  $\text{End}(V)$ , that is,

$$\rho : \mathcal{S} \rightarrow \text{End}(V), \quad (5.1)$$

where  $\text{End}(V)$  is a module of the endomorphisms of some  $\mathcal{R}$ -module  $V$ . Choosing a basis for  $V$  will give us matrices for  $\text{End}(V)$ . The various morphisms are defined by the commutative diagram illustrated in Figure 2. The

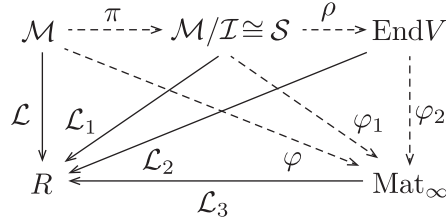


Figure 2: Commutative diagram showing the various homomorphism maps. The dotted arrows are ring homomorphisms whilst the solid arrows are module homomorphisms.

existence of  $\rho$  gives us the matrix product representation theorem.

**Theorem 10** (Matrix Product Representation). *Let  $\mathcal{S}$  be the shock ring (4.7a) (or isomorphically  $\mathcal{M}/\mathcal{I}$ ) and  $\hat{X}$  and  $\hat{Y}$  the matrices given by (3.35) and (3.36). If  $\varphi_1 : \mathcal{S} \rightarrow \text{Mat}_\infty$  (see Figure 2) is defined by  $\varphi_1(e_1) = \hat{X}$  and*

$\varphi_1(e_2) = \hat{Y}$ , then  $\varphi_1$  is an injective ring homomorphism. Furthermore  $\varphi$  is a matrix representation (ie.  $\varphi : \mathcal{M} \rightarrow \text{Mat}_\infty$  is a ring homomorphism) of  $\mathcal{M}$  with  $\ker(\varphi) = \mathcal{I}$ .

*Proof.* The proof is a direct check that  $\hat{X}\hat{Y} - \alpha\beta(\hat{X} + \hat{Y}) = 0$  and hence the matrices satisfy the defining relation, (4.1a), of  $\mathcal{I}$ . Thus  $\varphi(\mathcal{I}) = 0$  and  $\ker(\varphi) = \mathcal{I}$  which implies  $\varphi_1$  is injective. Note, we must have  $d = \infty$  (if  $d$  is finite then that implies exists  $n \leq d$  such that  $\hat{X}^n$  is a linear combination of powers of  $\hat{X}$  smaller than  $n$  (Caley-Hamilton theorem) which would imply that  $e_1^n$  is a linear combination of lower powers of  $e_1$  which is not the case).  $\square$

Since  $\varphi_1$  is an injective ring homomorphism any element of  $\mathcal{S}$  is uniquely represented by a sum/product of the  $\hat{X}$  and  $\hat{Y}$  matrices, for example, if

$$g = e_1^{\otimes 5} \otimes e_2^{\otimes 3} + e_1 \otimes e_2^{\otimes 2} \otimes e_1$$

then

$$\varphi(g) = \varphi_1(\pi(g)) = \hat{X}^5 \hat{Y}^3 + \hat{X} \hat{Y}^2 \hat{X}.$$

We note that  $\hat{X}$  and  $\hat{Y}$  are precisely the the  $D_3$  and  $E_3$  representations of [1] (with  $\kappa = \sqrt{\alpha\beta(\alpha + \beta - 1)}$ ). Thus we see that the third representation of [1] is directly related to the matrix of first moments with respect to the BiOPS given in Theorem 8.

We still need to determine how to evaluate  $\mathcal{L}$  on a matrix representation (see Figure 2) that is, what is  $\mathcal{L}_3$ ? This is given by the following theorem

**Theorem 11.** *Let  $(\mathbf{G}_{i,j})_{i,j \geq 0}$  be the matrix representation,  $\mathbf{G} = \varphi(g) = \varphi_1(\pi(g))$ , of some element  $g \in \mathcal{M}$  and  $\mathcal{L}_3$  the map defined by the commutative diagram Figure 2. Then*

$$\mathcal{L}_3(\mathbf{G}) = \mathbf{G}_{0,0} \tag{5.2}$$

or

$$\mathcal{L}(g) = \mathbf{G}_{0,0}. \tag{5.3}$$

Thus  $\mathcal{L}(g)$  is just the  $(0,0)$  element of the matrix representation of  $g$ . We give a proof following Theorem 12.

If we consider the matrix representations of  $P_n$  and  $Q_n$  ie. the matrices  $\mathbf{P}_n$  and  $\mathbf{Q}_n$ , they have bi-diagonal structure

$$(\mathbf{P}_n)_{i,j} = \begin{cases} 1 & \text{if } j = i + n \\ \alpha(\beta - 1) & \text{if } j = i + n - 1 \text{ and } j \geq n \\ 0 & \text{otherwise.} \end{cases} \quad (5.4)$$

The matrix representation of  $Q_n$  has a similar, but transposed, form

$$(\mathbf{Q}_n)_{i,j} = \begin{cases} 1 & \text{if } i = j + n \\ \beta(\alpha - 1) & \text{if } i = j + n - 1 \text{ and } i \geq n \\ 0 & \text{otherwise.} \end{cases} \quad (5.5)$$

This structure of the matrices  $\mathbf{P}_n$  and  $\mathbf{Q}_n$  give

$$(\mathbf{P}_n \mathbf{A} \mathbf{Q}_m)_{0,0} = \mathbf{A}_{n,m}. \quad (5.6)$$

for any infinite matrix  $(\mathbf{A}_{i,j})_{i,j \geq 0}$ . Since  $\mathcal{L}(P_n g Q_m) = (\mathbf{P}_n \mathbf{G} \mathbf{Q}_m)_{0,0}$  we get the following generalisation of Theorem 11.

**Theorem 12** (Matrix-Moment). *Let  $g \in \mathcal{M}$ . Then*

$$\mathcal{L}(P_n g Q_m) = \mathbf{G}_{n,m}.$$

where  $\mathbf{G} = \varphi(g) = \varphi_1(\pi(g))$  is the matrix representation of  $g$ .

Thus the matrix elements of  $\mathbf{G}$  can be computed either by substituting  $\hat{X}$  and  $\hat{Y}$  wherever  $e_1$  and  $e_2$  appear in  $g$ , or by evaluating  $\mathcal{L}(P_n g Q_m)$ . Conversely, knowing the matrix elements of  $\mathbf{G}$  (obtained via substituting  $\hat{X}$  and  $\hat{Y}$ ) gives the values of  $\mathcal{L}(P_n g Q_m)$ .

*Proof of Theorem 11.* We use the condition that  $\mathcal{L}_3$  is an  $\mathcal{R}$ -module morphism and the two equations (2.2b) and (2.2c) in conjunction with the standard elementary matrix  $E_{i,j}$  basis for  $\text{Mat}_\infty$ . We show how  $\mathcal{L}_3$  evaluates on a basis element of  $\mathcal{M}$ . Let  $\mathbf{A}$  be the matrix representation of an arbitrary basis element  $b \in \mathcal{M}$ ,  $\hat{X}$  the matrix representation of  $e_1$  and  $\hat{Y}$  the matrix representation of  $e_2$ . Then, from (2.2b),

$$\begin{aligned} \mathcal{L}_3(\mathbf{A} \hat{X}) - \alpha \mathcal{L}_3(\mathbf{A}) &= 0 \\ \mathcal{L}_3(\mathbf{A}(\hat{X} - \alpha \mathbf{I})) &= 0 \end{aligned}$$



where  $\mathbf{I}$  is the identity matrix. Expanding in the basis  $E_{ij}$  gives

$$\begin{aligned} \sum_{i,j \geq 0} \mathcal{L}_3 \left( \sum_{k \geq 0} A_{i,k} (\hat{X}_{k,j} - \alpha \delta_{k,j}) E_{ij} \right) &= 0 \\ \sum_{i,j \geq 0} \left( \sum_{k \geq 0} A_{i,k} (\hat{X}_{k,j} - \alpha \delta_{k,j}) \right) \mathcal{L}_3(E_{ij}) &= 0. \end{aligned} \quad (5.7)$$

Thus we look for any non-trivial values of  $\mathcal{L}_3(E_{ij})$  that will satisfy (5.7) for an *arbitrary* basis element  $b$  of  $\mathcal{M}$ . We first try make each  $i, j$  term vanish. Substituting from (3.35) gives two cases arising from the coefficients of  $\mathcal{L}_3(E_{ij})$ :

$$\text{Case: } i = j = 0: \quad (\alpha A_{0,0} - \alpha A_{0,0}) \mathcal{L}_3(E_{ij}) = 0 \quad (5.8)$$

$$\text{Case: } i, j \text{ not both 0:} \quad (\alpha \beta (A_{i,j} - A_{i,j+1}) - \alpha A_{i,j}) \mathcal{L}_3(E_{ij}) = 0 \quad (5.9)$$

These equations have a non-trivial solution (for arbitrary  $A$ ) if we take  $\mathcal{L}_3(E_{0,0})$  arbitrary but for  $i$  and  $j$  not both zero we must have  $\mathcal{L}_3(E_{i,j}) = 0$ . Since  $\mathcal{L}(1) = 1$  we need to take  $\mathcal{L}_3(E_{0,0}) = 1$ . For this choice we get  $\mathcal{L}_3(M) = \sum_{i,j \geq 0} A_{i,j} \mathcal{L}_3(E_{i,j}) = A_{0,0} \mathcal{L}_3(E_{0,0}) = A_{0,0}$ . The same result is obtained from the second equation (2.2c) and hence is consistent.  $\square$

## 6 The Second Moment and Classical Orthogonal Polynomials

We now consider the matrix  $\mathbf{W}$  defined by the second moment,

$$\mathbf{W}_{n,m} = \mathcal{L}(\hat{P}_n e_1 e_2 \hat{Q}_m) \quad n, m \in \mathbb{N}_0. \quad (6.1)$$

Combining the matrix representation Theorem 10 and the matrix-moment Theorem 12 we get that  $\mathbf{W}$  is tri-diagonal.

**Theorem 13.** *The second moment  $\mathbf{W}_{n,m} = \mathcal{L}(\hat{P}_n e_1 e_2 \hat{Q}_m)$  is given by the symmetric tridiagonal matrix*

$$\mathbf{W} = \alpha \beta \begin{pmatrix} \alpha + \beta & \sqrt{\alpha \beta (\alpha + \beta - 1)} & & & \\ \sqrt{\alpha \beta (\alpha + \beta - 1)} & 2\alpha \beta & \alpha \beta & & \\ & \alpha \beta & 2\alpha \beta & \alpha \beta & \\ & & \alpha \beta & 2\alpha \beta & \alpha \beta \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \quad (6.2)$$

Thus, associated with the second moment  $\mathcal{L}(\hat{P}_n e_1 e_2 \hat{Q}_m)$  is the classical (ie. satisfying a three term recurrence relation) sequence of Chebychev-like polynomials  $\{T_n(x)\}_{n \geq 0}$ , satisfying

$$W_{n,n-1}T_{n-1} + (2W_{n,n} - x)T_n + W_{n,n+1}T_{n+1} = 0 \quad (6.3)$$

with initial values  $T_0 = 1$  and  $T_{-1} = 0$ . Since

$$\mathbf{W} = \hat{\mathbf{X}}\hat{\mathbf{Y}}.$$

in this context the classical orthogonal tridiagonal matrix is seen to factorise into a product of two bi-diagonal matrices. This is a form of LU-decomposition of the tridiagonal matrix into upper and lower triangular matrices.

## 7 Concluding Remarks

We have shown that we can get the third matrix representation of [1] by using a pair of bi-orthogonal polynomial sequences defined via a bi-moment matrix (3.2) and a linear form (1). The bi-moment matrix is constructed using the linear form and the relation  $e_1 \otimes e_2 = \alpha\beta(e_1 + e_2)$ . The ideal generated by the relation defines a quotient ring (the shock ring). In this ring the “bulk” terms in the master equation satisfy the stationarity condition (but the “boundary” terms do not). For the shock ring there exists an injective ring homomorphism to infinite dimensional matrices (Theorem 10). This tells us a matrix representation exists but not directly what the matrix elements might be.

To find the matrix elements, we show that the matrix elements arise through the matrices defined by the first moments of the linear form  $\mathcal{L}$  with respect to a (normalised) bi-orthogonal pair of sequences of polynomials  $\{P_n\}$  and  $\{Q_n\}$ . In particular the two matrices  $\hat{X}_{n,m} = \mathcal{L}(\hat{P}_n e_1 \hat{Q}_m)$  and  $\hat{Y}_{n,m} = \mathcal{L}(\hat{P}_n e_2 \hat{Q}_m)$  satisfy the diffusion algebra (Theorem 7).

The form of bi-orthogonality that arises in this paper doesn’t appear to be directly connected with other forms on bi-orthogonality, such that arising via integration over the unit circle [9] or that for Laurent polynomials [10] but appears to have a close relationship to the LU decomposition of matrices.

Is this form of bi-orthogonality only useful in the open boundary TASEP model or is it of wider applicability? We do have preliminary results that show the polynomial sequences generalise to the three parameter ASEP ( $\alpha$ ,  $\beta$  and  $q$  non-zero) where  $q$ -bi-orthogonal sequences are obtained.

Another point of interest is, does this bi-orthogonality give any new insight into classical (ie. three term recurrence) orthogonal polynomials? It would be of interest to know if, for example, given any classical orthogonal polynomial sequences  $\{T_n\}$  does there exist a linear map  $\mathcal{F}$  and a pair of sequences  $\{A_n\}$  and  $\{B_n\}$  of bi-orthogonal polynomials with respect to  $\mathcal{F}$  such that the second moment gives the recurrence for  $\{T_n\}$ ? Since the five parameter ASEP is connected [5, 11] with the Askey-Wilson polynomials [12] it would suggest this is the case.

Finally, what about the combinatorics of this formalism? The connection between classical orthogonal polynomials and the combinatorics of lattice paths is well established [13, 14] as is the combinatorics of the ASEP model [?]. Clearly the bi-diagonal structure of the  $\bar{X}$  and  $\bar{Y}$  matrices connect to binomial lattice paths (aka. fully directed paths) and the tridiagonal matrix  $W_{n,m} = \mathcal{L}(P_n e_1 e_2 Q_m)$  to Motzkin paths. However, in this instance there is no Hankel matrix of moments – it is replaced by the bi-moment matrix. Preliminary studies suggest we still have a Gessel-Viennot set of non-intersecting paths [15] associated with the bi-moment matrix analogous to the situation with the Hankel matrix.

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